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Some Equivalence Classes of Polygonal 7-Chains

Paul Wayne Lewis
University of Tennessee - Knoxville

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Name: PAUL LEWIS

College: ARTS & SCIENCES Department: MATHEMATICS

Faculty Mentor: R J DAVERMAN

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I have reviewed this completed senior honors thesis with this student and certify that it is a project commensurate with honors level undergraduate research in this field.

Signed: Robert J Daverman, Faculty Mentor

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Some Equivalence Classes of Polygonal 7-chains

Paul Lewis

A polygonal chain is the result when one segment of a polygonal knot is removed. When the number of segments in a chain and the lengths of the segments are held constant, equivalence classes of knotted polygonal chains can be determined. A chain is said to be knotted if there is no isotopy from the chain to the trivial chain. Whether or not a chain is knotted depends on the relative lengths of the segments, therefore; the word *possible* will often precede knotted chain in this paper to indicate that the relative length requirements must be met. Two chains belong to the same equivalence class if there is an isotopy from one chain to the other. Cantarella and Johnston showed that no polygonal n-chain with $n \leq 4$ segments is knotted. They also determined all of the possible equivalence classes for 5-chains. Clark and Venema continued this work by determining all possible equivalence classes of 6-chains. The purpose of this paper is to briefly examine some of the possibilities in dealing with polygonal chains with seven segments.

In examining chains with seven segments, it is immediately obvious that many generalizations from the $n = 6$ case can be made, as several generalizations from the $n = 5$ case were found for polygonal chains with six segments. While forming these equivalence classes is very simple, counting how the number of such classes is a little tedious. Remaining consistent with previously named chains, we find inner trefoils, outer trefoils and inner and outer stuck trefoils, all of which can be left or right. Thus, by simply observing that we may add a segment to any pre-existing trefoil chain with six segments, we have already determined several equivalence classes that may exist for some given lengths l_1, l_2, ... l_7. The following is a list of equivalence classes that could arise. There are 36 trefoil equivalence classes listed.

6 *outer trefoils*. There are two 5-chain trefoils to which two additional segments could be added to the ends in three different ways.

2 *inner trefoils*. There are two 6-chain inner trefoils to which another segment could be added to the middle.

4 *inner/outer trefoils*. There are two 6-chain inner trefoils to which a segment could be added to either of the two ends.

2 *inner stuck trefoils*. These are the two stuck trefoils with an added or divided inner segment.

2 *inner reverse stuck trefoils*. These are the two reverse stuck trefoils with an added or divided inner segment.

4 *outer stuck trefoils*. There are two stuck trefoils to which segments could be added to either end.

4 *outer reverse stuck trefoils*. There are two reverse stuck trefoils to which segments could be added to either end.

2 *doubly stuck trefoils.* These are stuck trefoils to which another segment has been added to the "not stuck end" so that the segment goes back through the chain causing the first and last segments to be stuck.

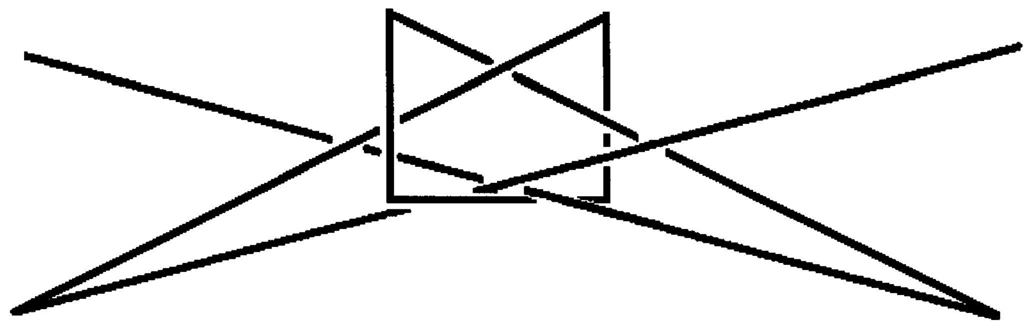

Figure 1 a doubly stuck trefoil 7-chain

2 *reverse doubly stuck trefoils.* These are reverse stuck trefoils to which another segment has been added to the "not stuck end" so that the segment goes back through the chain causing the first and last segments to be stuck.

8 *"back through" trefoils.* These chains are stuck or reverse stuck trefoils with segments added to the "stuck end" that come back through the chain. There are two different equivalence classes for each of the 2 stuck trefoils and 2 reverse stuck trefoils. Unlike with the 6-chain stuck trefoils, the 7-chain "back through"

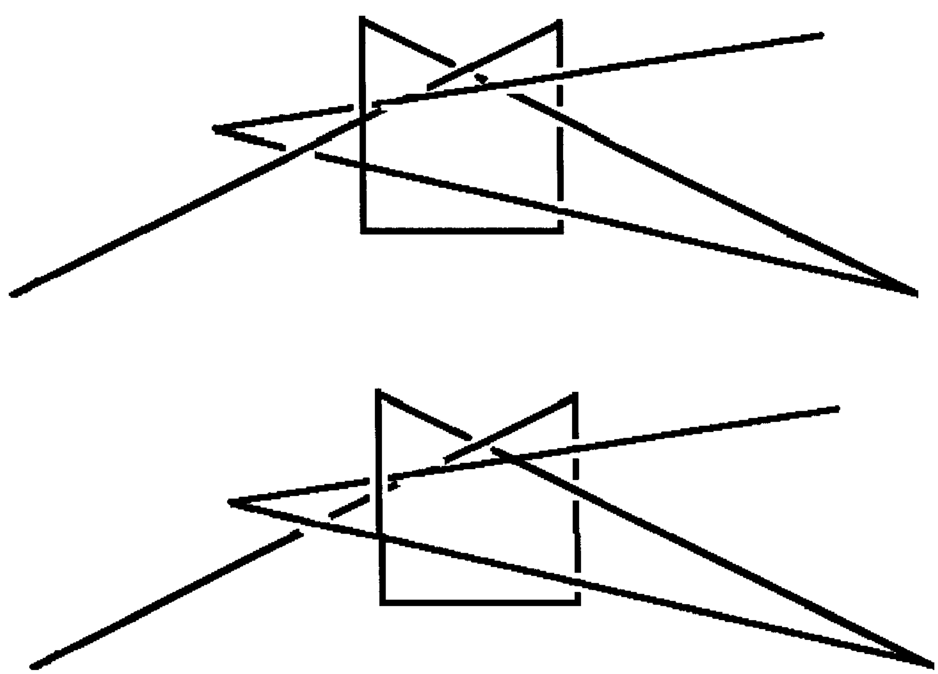

Figure 2 different types of "backthroughs"

trefoils' equivalence class depends on where the final segment comes back through the chain. The figure above shows representatives of two different equivalence classes when the non-stuck end segment is relatively long.

In our running total of equivalence classes, we have as yet failed to mention modifications of figure-eight chains. Figure eight 7-chains are much easier to count simply because a basic figure-eight chain requires six segments, leaving less room for variation than in trefoil chains, which require a minimum of five segments. Considering ordinary and special figure-eight chains and their mirror images, we find there may exist numbers of classes up to:

2 *outer ordinary figure-eight chains* and 2 *reverse outer ordinary figure-eight chains*. These are ordinary figure eight chains with a segment added to one of the ends.

2 *outer special figure-eight chains* and 2 *reverse outer special figure-eight chains*. These are special figure eight chains with a segment added to one of the ends.

2 *inner ordinary figure-eight chains*. These are ordinary figure-eight chains with a divided inner segment.

2 *inner special figure-eight chains*. These are special figure-eight chains with a divided inner segment.

2 *stuck ordinary figure-eight chains* and 2 *reverse stuck ordinary figure-eight chains*. These are ordinary figure eight chains with a segment added to one of the ends which goes back through the chain.

2 *stuck special figure-eight chains* and 2 *reverse stuck special figure-eight chains*. These are ordinary figure eight chains with a segment added to one of the ends which goes back through the chain.

Thus, by simply adding segments to the previously identified equivalence classes, we were able to determine 56 possible equivalence classes. So including the trivial chain, we have determined 57 equivalence classes, about 4 times the total maximum number of equivalence classes for polygonal 6-chains. The relative segment lengths required so that these chains are knotted are easily calculated from the bounds on the six-segment case.

One can clearly see that we are not yet finished counting equivalence classes. One would expect the existence of equivalence classes that occur with 7-chains that could not occur with six. I propose that it might be useful, as the number of segments in chains increases, to have some sort of terminology to characterize chains of this kind and also to characterize chains that are derivatives of knotted chains with fewer segments, an indicator of how many segments are essential to the knotting of the chain. Because of the expected exponential growth of the number of possible equivalence classes of chains, it may be appropriate to look at particular subsets of the set of equivalence classes of n-chains with fixed segment lengths. While it almost may not the best of way of attacking the proposed task, the following is an example of an attempt to do so. I will now make a few definitions:

Definition. An *n-essential chain of order m* is a knotted polygonal p-chain such that m is the largest number of segments that can be removed from either of the ends of the chain or from the middle of the chain (where the chain is reconnected if it is an inner case) such that the remaining chain remains knotted and n + m = p.

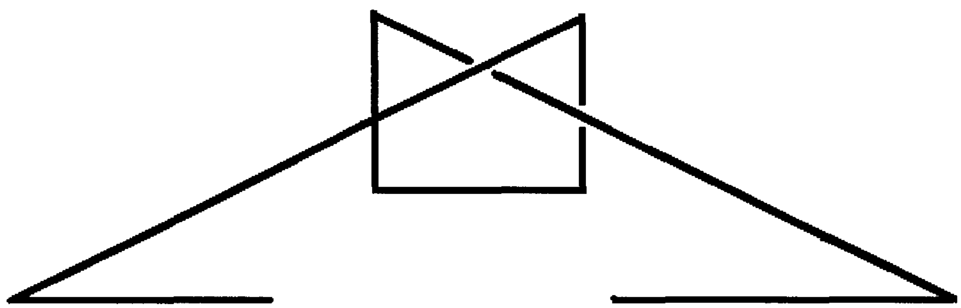

Figure 3 an outer trefoil chain that is 5-essential order 2

Definition. An *n-essential chain* is an n-essential chain of order m where m ≥ 0.

Definition. A *chain of order m* is an n-essential chain of order m with n ≥ 5.

Definition. A *prime chain* is any n-essential chain of order 0.

Using the word prime in the last definition seems reasonable at first glance. It is obvious that a chain that satisfies the above could be broken down to form a smaller knotted structure, but this definition may prove to be to be inappropriate or problematic in the future.

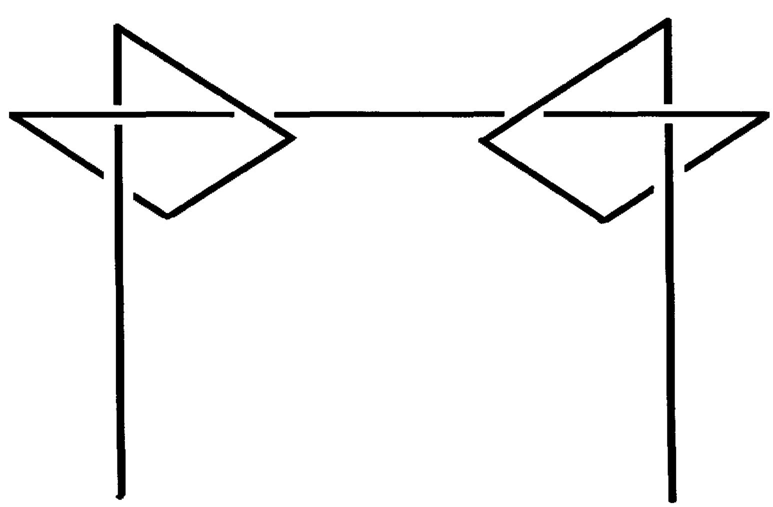

Figure 4 a "composite" 9-chain

This is probably the simplest way to construct what we might think of as a "composite chain". In particular, we have two trefoil chains joined together in the simplest possible way with one end segment performing as the beginning segment of the other trefoil part. The chain is clearly not prime, because we can remove four segments (one of the trefoil parts) and the remaining chain will still be knotted. So if Figure 4 is the first chain we encounter that we would think of as composite, and Figure 3 above gave us an example of a 7-chain that is not prime nor the trivial chain, we may have already encountered a problem. It is easily seen why this occurs. If we add a prime knot with the trivial knot, we obtain the prime knot again, but this is not the case with chains. There are many different trivial chains, in fact infinitely many, and adding any one of them to a prime chain will necessarily yield a chain that is no longer prime but also not "composite".

Essentially, up until this point we have only looked at non-prime polygonal 7-chains. We now attempt to find the prime 7-chains.

The nontrivial chains with five segments were labeled trefoil chains because if the two ends of the chain were connected by a sixth segment the result would be a polygonal trefoil knot, a knot with three crossings. In the same way, six segment chains that were not generalizations of five segment chains were called figure eight chains after the four crossing figure eight knot. This leads to the question, do there exist polygonal 7-chains such that connecting the ends yields a five crossing knot? Any polygonal n-chain can be made into a polygonal knot of n + 1 segments. If the end segments are made long enough, there is always a line that can connect the first and last vertices of the chain given that the first and last segments do not lie on a line in space, and if they do, we assume there is enough wiggle room in one of the end segments so that this can be corrected. The answer

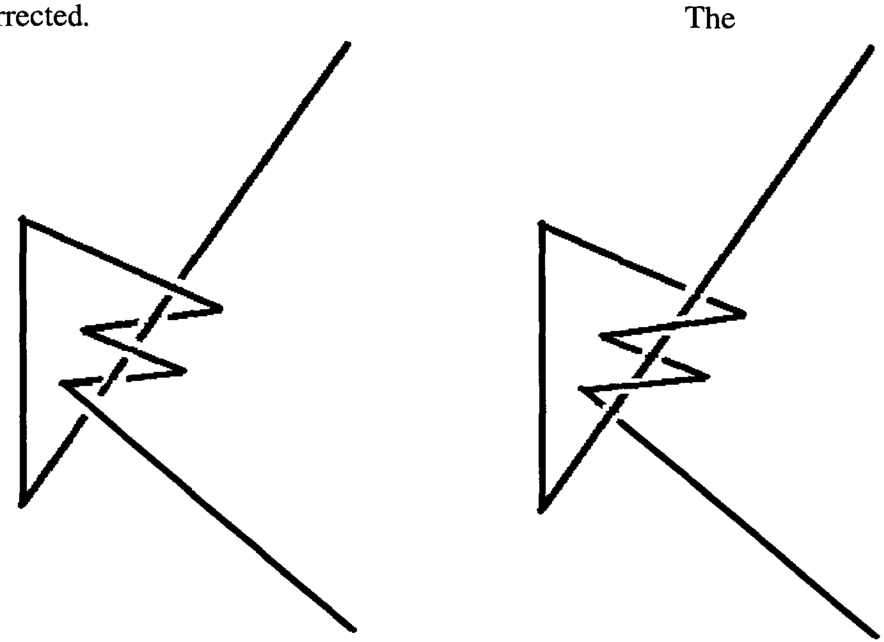

Figure 5 a 5_1 chain and its mirror image

is yes. There are such 7-chains, and two such chains are depicted in Figure 5. The chains are formed using a construction for (n, 2) torus knots with odd $n \geq 3$ from a previous paper co-written with Charlie Wiseman on stick numbers of torus knots with sixteen or fewer crossings. Observe that when one segment of the (5,2) torus knot, also known and from here on referred to as the 5_1 knot, is deleted the result is a knotted 7-chain when the first and last segments are "long enough". The two chains in Figure 5 will be referred to as 5_1 chains. When attempting to unravel a potential 5_1 chain, it becomes clear that "long enough" is always satisfied if $x_1 > x_2 + x_3 + x_4 + x_5 + x_6$ and $x_7 > x_2 + x_3 + x_4 + x_5 + x_6$ where x_i is the length of the ith segment. After sliding the rest of the chain along one of the end segments, as to directly pull the end segment from the rest of the chain, other inner segments can be folded outward, one at a time, until the other end segment is reached. The inner segments of the chain are flattened to allow the maximum amount of room for the end segment to pass through the space. Here it becomes apparent that the end segment should be longer than sum of the interior segments.

Thus we have determined two possible equivalence classes that correspond to the 5_1 knot. Now we will look at a couple of chains (Figure 6) that correspond to the other five crossing knot, the 5_2 knot. The Appendix shows the isotopies that change the closed chains into the 5_2 knot or its mirror image. Finding segment length requirements for these chains to be knotted is more difficult than with the previous chains. Looking at the chain pictured at the right of the figure, we will call the end at the left segment 1 and the end segment on the right segment 7. While other segments lengths may be involved in the exact inequalities, it appears true that the chain cannot be undone starting with segment 7 if $x_7 > x_4 + x_5 + x_6$. To prevent unraveling starting with segment 1, on the other hand, it must be true that $x_1 > x_2 + x_3 + x_4 + x_5 + x_6$. To see this, bring vertex 2 (the vertex joining segments 1 and 2) to the other side of vertex 7 (the vertex joining segments 6 and 7) and turn segment 1 so that it is almost parallel to segment 5. Continue pulling vertex 2 and swing it around until it is almost parallel with segment 7. We can then easily see why the chain is knotted if the inequality holds. Of course, if segments 2, 3, and 4 are relatively short, we may not even be able to bring vertex 2 past vertex 3, but this inequality gives a worst case scenario for when the chain is knotted. Similarly, small x_2 and x_3 may allow for smaller x_7, but it appears that our inequalities are sufficient, in any case.

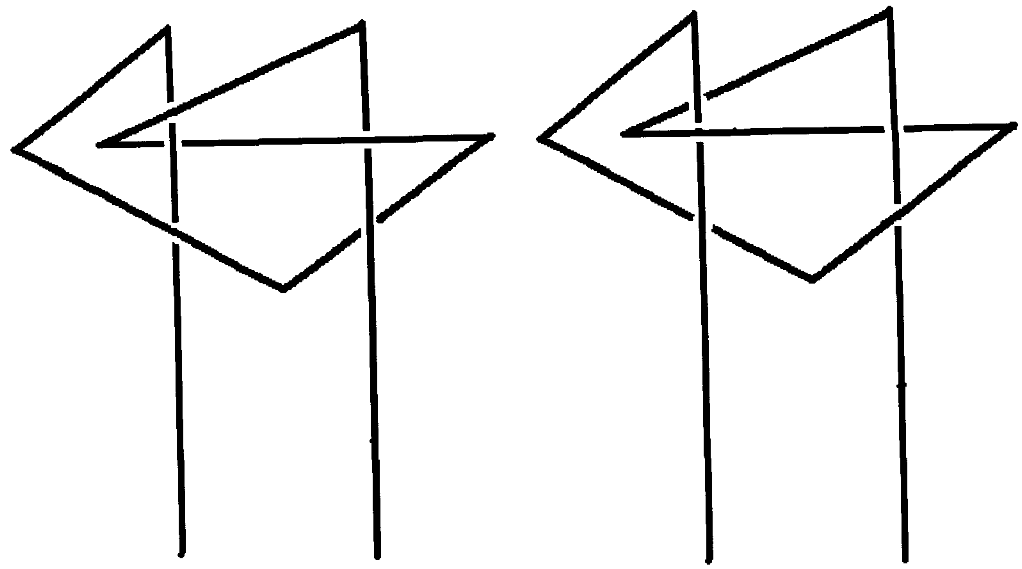

Figure 6 a 5_2 chain and its mirror image

Recall that there are two different types of figure-eight 6-chains. Therefore, it is reasonable that there may be more than one type of each 5_1 7-chains and 5_2 7-chains.

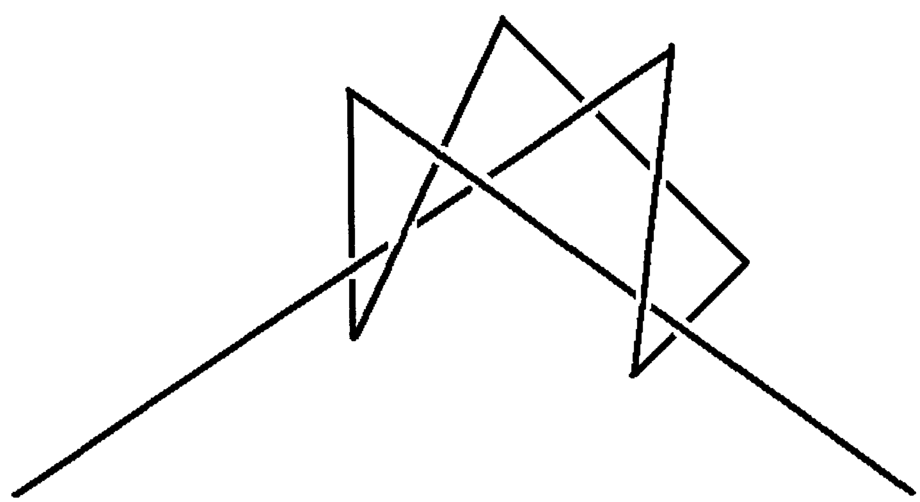

Figure 7 a chain that cannot exist

In Figure 7 we will call the end segment at the left segment one and the end segment at the right segment 7. At first the projection appears to be a likely candidate for a chain that corresponds to either the 5_1 knot or the 5_2 knot, as it was constructed by taking a trefoil chain and adding two segments to the chain's interior. But upon adding the closing segment and trying to transform the knot into the 5_1 or the 5_2 , it was discovered that the knot is actually a six crossing knot. As we are about to show, this cannot happen. Even though it was not obvious from the projection or even from a model that was constructed, the chain depicted it the projection cannot exist.

If K_1 is a polygonal knot formed by adding a closing segment to a chain, and K_2 is the smooth, non-polygonal version of K_1, then $c(K_2) + 3 \leq S$, where c is the crossing number and S is the number of segments in K_1.

This is a direct result of the fact that for any given knot K, $c(K) + 3 \leq s(K)$, where s(K) is the stick number of the knot. Since K_1 and K_2 are topologically equivalent and S is an upper bound for $s(K_2)$, we have $c(K_2) + 3 \leq s(K_2) \leq S$. In our example above $c(K_2) + 3 = 9 \leq 8 = S$, which is contradiction. Thus, the chain cannot exist. In further examination of the model that was constructed, it became apparent that the reason the chain does not exist is that the projection requires segment five to be slightly bent. The relative frequency of how helpful the above inequality can be in determining whether or not a chain can exist as depicted in a projection may be low; after all, one must be able to compute the crossing number of the corresponding knot. But, in this case, the inequalities proved to be helpful and may prove to be helpful in similar situations.

The above projection did not result in complete failure. By changing one of the projection's crossings, as shown below, one obtains a projection of a legitimate chain. By deforming the corresponding knot as show in the Appendix, we see that the chain corresponds to the 5_1 knot. Thus, this equivalence class, which is clearly not the same as the previous 5_1

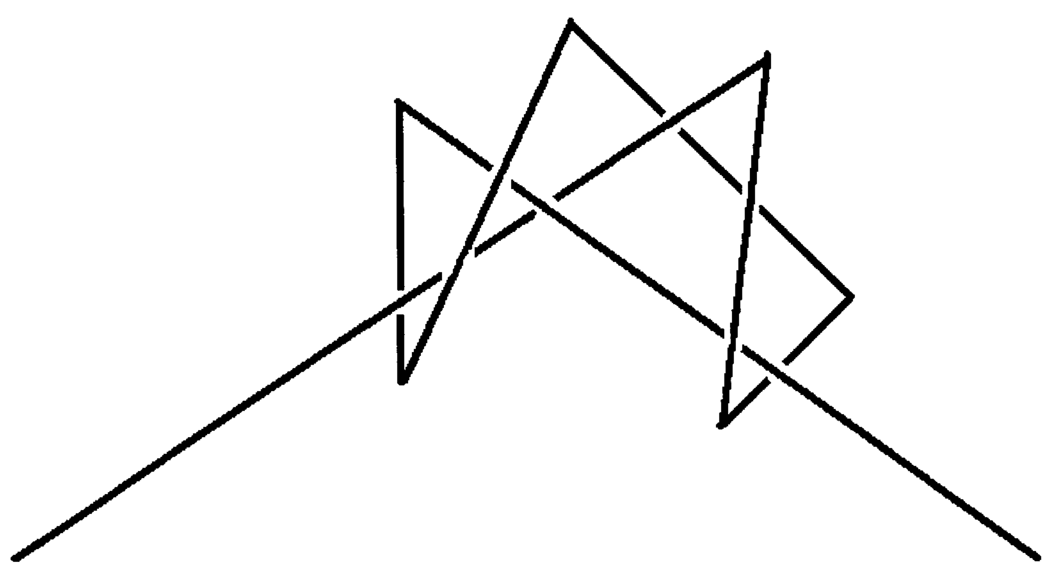

Figure 8 a projection of a 5_1b chain

class, will be referred to as the class of 5_1b chains. The chain in Figure 9 is another type of 5_2 chain, we will call this equivalence class 5_2b. Both of these classes are knotted if the two ends are longer than the remaining segments, but much more precise relations are expected to exist.

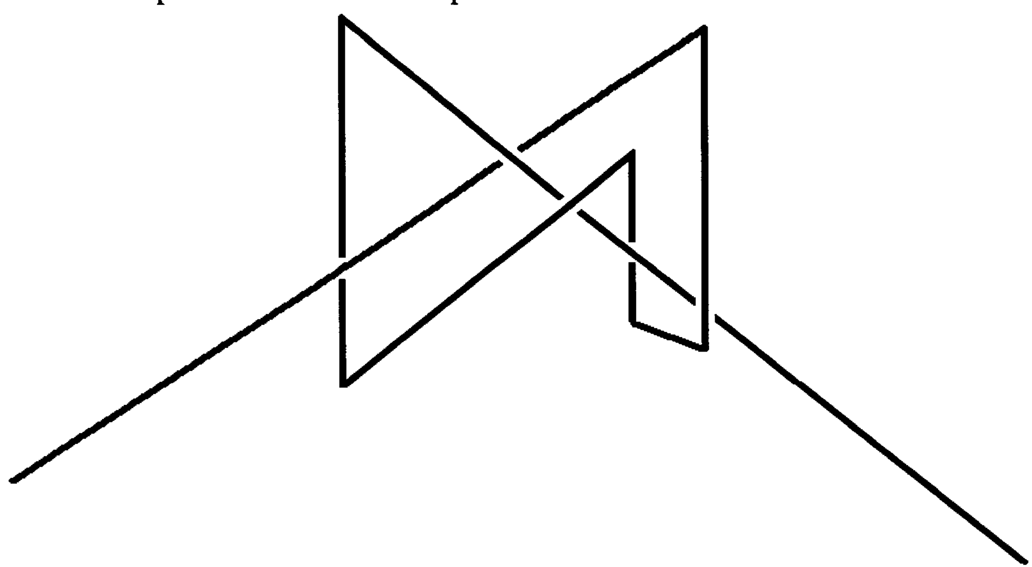

Figure 9 a projection of a 5_2b chain

With each of the four classes of prime chains we have identified and the classes of their mirror images, we have determined the maximum number of equivalence classes of 7-chains is at least 65. The maximum number of equivalence classes is expected to grow exponentially as n increases. The sharp increase from 13 to at least 65 seems to support this conjecture. There may be several more equivalence classes that we have not determined. Clark and Venema completely determined the equivalence classes of 6-chains using the radial projection function. Using the radial projection function for chains with seven or more segments would take a very long time. At one point in determining 6-chains, there were 137 cases that eventually lead to 13 equivalence classes. Considering there are at least 65 7-chain classes, the number of cases using the radial projection function would be very large. In the future, it might be helpful to develop computer programs to determine classes, or find a way to group equivalence classes and determine which groups are most important for study. Also, developing devices similar to invariants in knots might be useful although they would most likely be very complex in definition.

5_2 Chain → 5_2 Knot

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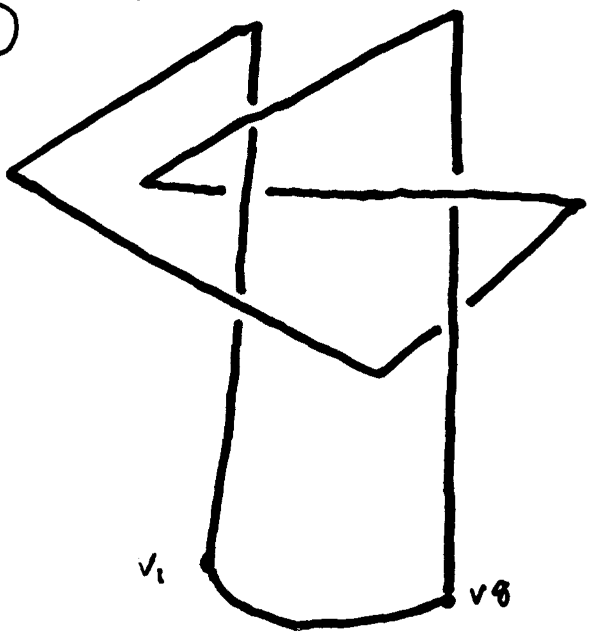

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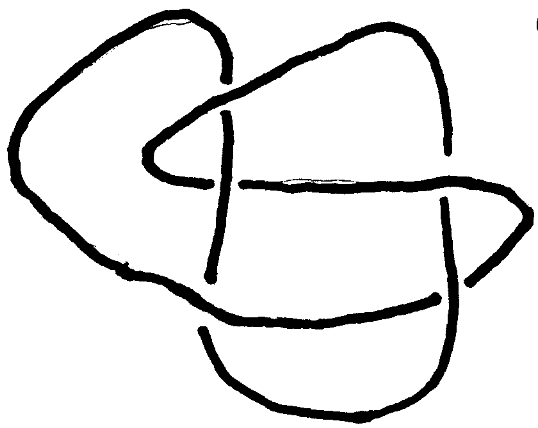

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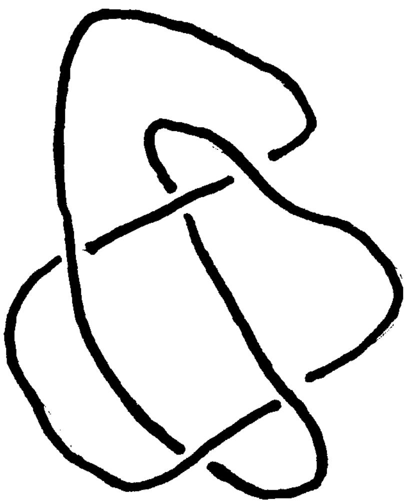

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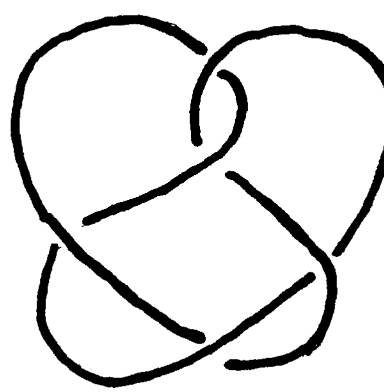
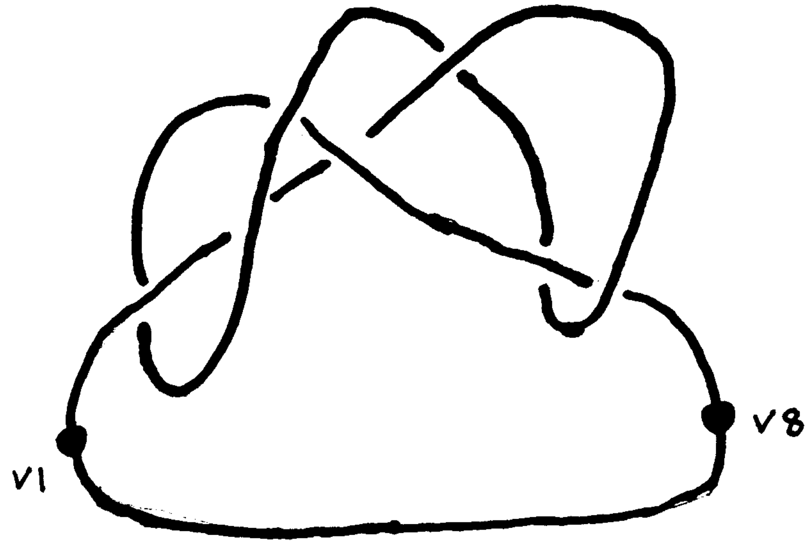


5_2 knot

5_1b Chain $\longrightarrow$ 5_1 Knot



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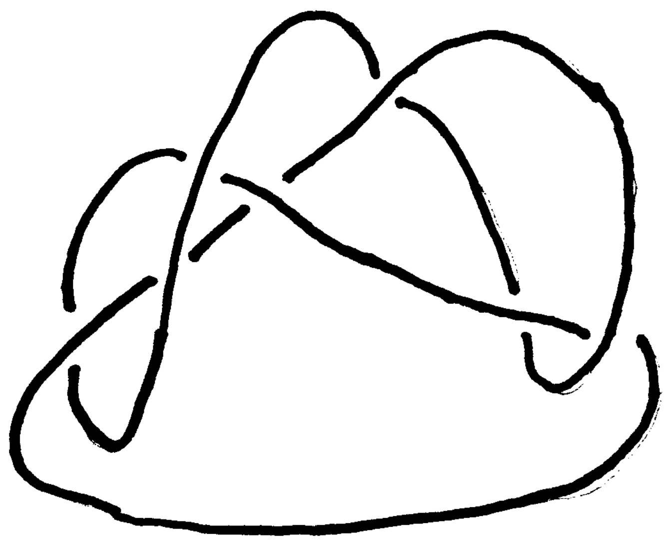

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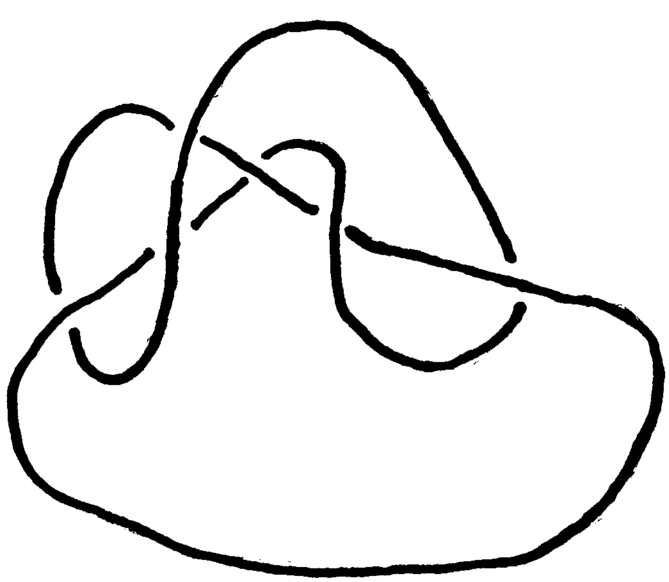

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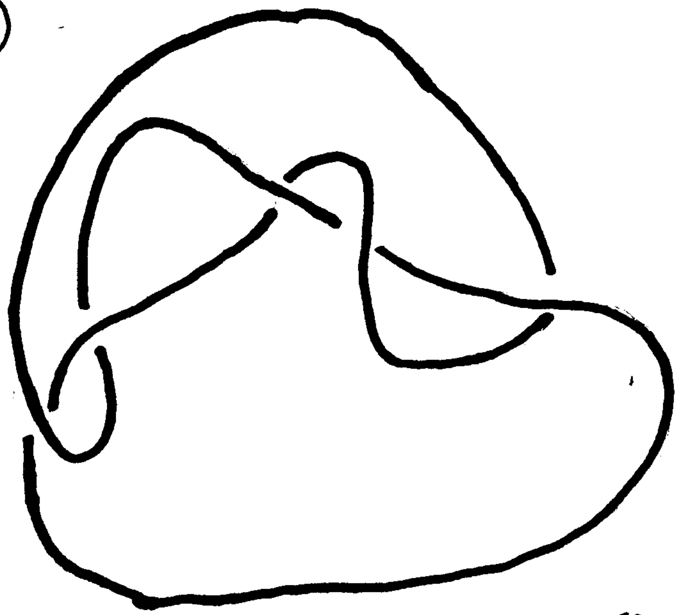

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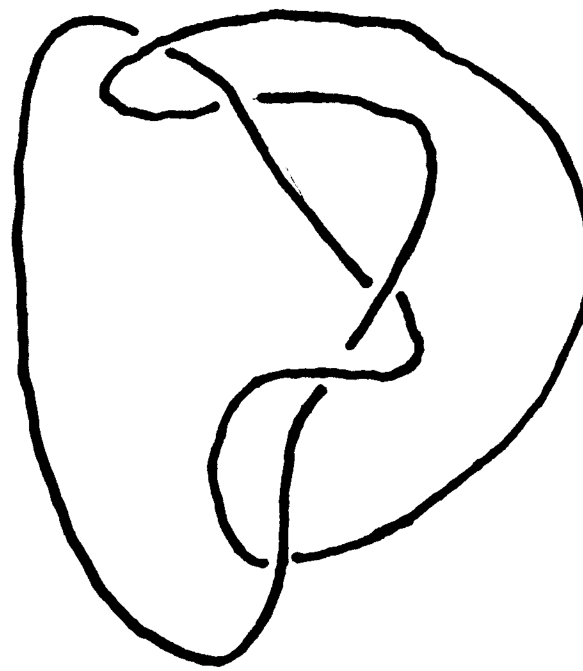

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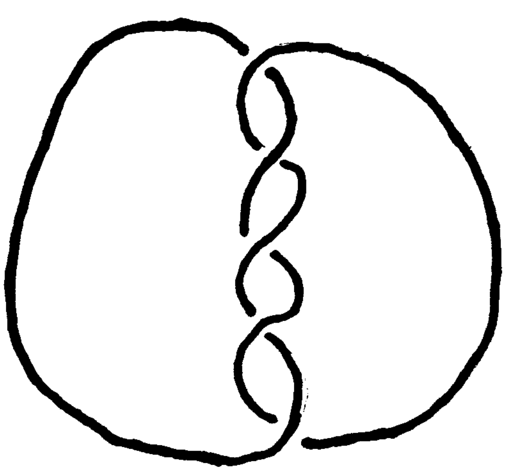

5_1 (or (5,2) Torus Knot)

5_2b Chain → 5_2 Knot

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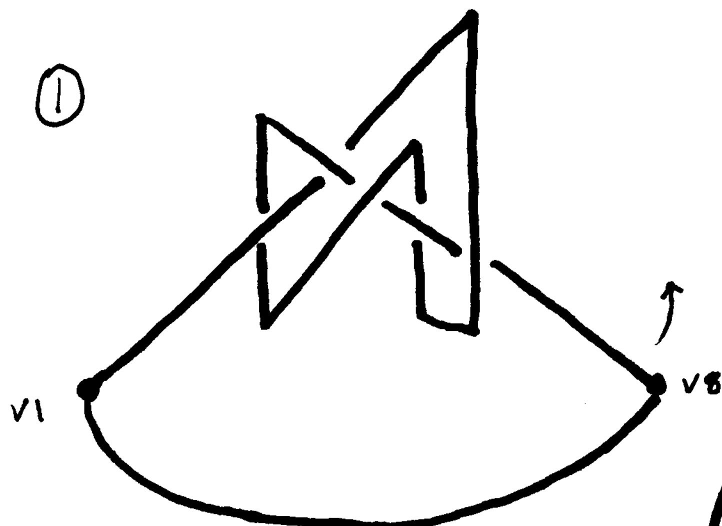

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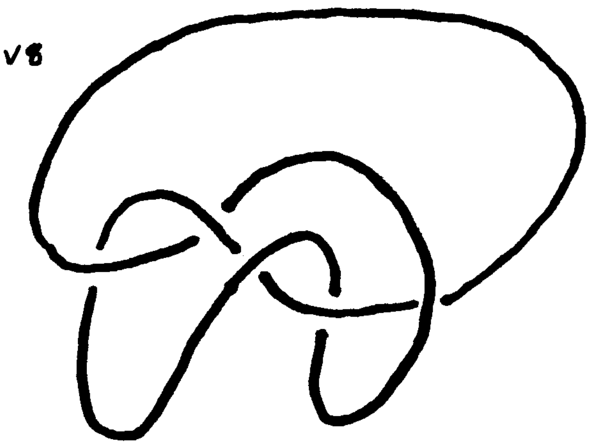

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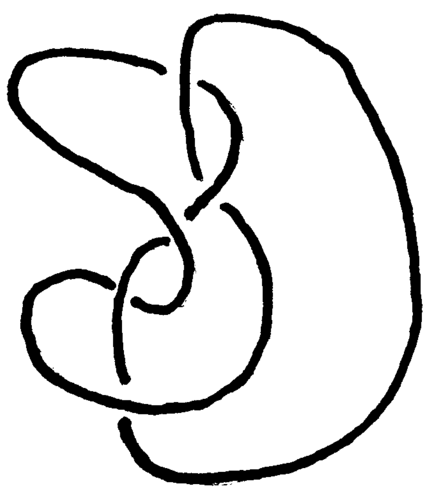

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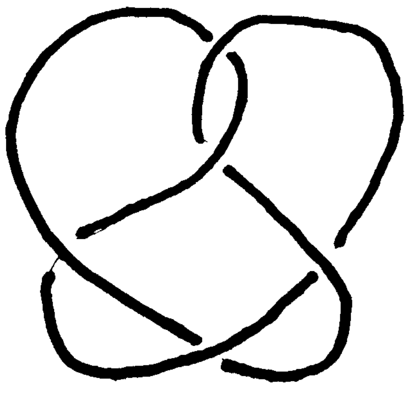

5_2 knot

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